

Problem Set: Human Capital and the Lucas Model

Advanced Macroeconomics — Dr Lei Pan — Total: 100 Marks

Instructions. Answer all questions. Show all mathematical derivations clearly. Answers without derivation receive limited credit. Unless otherwise stated, assume all variables are strictly positive and all parameters lie in economically meaningful ranges. The problem set is based on the augmented Solow model with human capital and the Lucas human-capital accumulation model.

Question 1: Human Capital, Conditional Income Gaps, and Augmented Solow Dynamics [Total: 45 marks]

A major empirical fact in growth economics is that cross-country income differences are strongly correlated with schooling, but measured education alone does not explain all international income gaps. Consider an augmented Solow–Swan economy in which human capital enters production through effective labour:

$$Y_t = K_t^\alpha [A_t H_t]^{1-\alpha}, \quad 0 < \alpha < 1,$$

where

$$H_t = L_t G(x), \quad G'(x) > 0, \quad G''(x) < 0.$$

Here x is average years of schooling, assumed constant over time for a given country. Technology and population evolve as

$$A_{t+1} = (1 + g)A_t, \quad L_{t+1} = (1 + n)L_t,$$

and capital accumulation is

$$K_{t+1} = (1 - \delta)K_t + sY_t, \quad 0 < s < 1, \quad 0 < \delta < 1.$$

Define

$$k_t \equiv \frac{K_t}{A_t L_t G(x)}, \quad D \equiv (1 + n)(1 + g), \quad b \equiv D - (1 - \delta).$$

- (a) Derive the intensive-form production function

$$\tilde{y}_t \equiv \frac{Y_t}{A_t L_t G(x)}$$

and the exact law of motion for k_t . Then derive $k_{t+1} - k_t$ and interpret the break-even investment term. [10 marks]

- (b) Solve for the positive steady-state capital stock k^* and output per worker $(Y_t/L_t)^*$. Assume $G(x) = e^{\eta x}$, where $\eta > 0$. Derive the predicted relative output per worker of country i relative to country j :

$$\frac{(Y/L)_i^*}{(Y/L)_j^*}.$$

Explain why this is a conditional-convergence prediction. [12 marks]

- (c) Suppose country R is a rich economy and country P is a poor economy. They share

$$\alpha = \frac{1}{3}, \quad \eta = 0.08, \quad g = 0.02, \quad \delta = 0.05,$$

but differ in saving, population growth, and schooling:

$$s_R = 0.28, \quad n_R = 0.01, \quad x_R = 12,$$

$$s_P = 0.12, \quad n_P = 0.025, \quad x_P = 7.$$

Using the exact formula for b , compute

$$\frac{(Y/L)_R^*}{(Y/L)_P^*}.$$

Decompose the ratio into the education component and the capital-deepening/demography component. [10 marks]

- (d) Prove local convergence to the steady state. Derive the convergence coefficient

$$\lambda = \mathcal{G}'(k^*).$$

Explain why countries with the same structural parameters but lower initial k_0 grow faster during transition. [8 marks]

- (e) Suppose schooling changes over time. Derive the growth-accounting equation for output per worker:

$$\Delta \ln(Y_t/L_t) = \alpha \Delta \ln(K_t/L_t) + (1 - \alpha) \Delta \ln A_t + (1 - \alpha) \Delta \ln G(x_t).$$

If $G(x) = e^{\eta x}$, rewrite the human-capital contribution in terms of Δx_t . [5 marks]

Question 2: The Lucas Human-Capital Model and Endogenous Growth

[Total: 55 marks]

Consider the Lucas economy. A representative agent has one unit of time each period. The fraction φ_t is used in goods production, while $1 - \varphi_t$ is used in education. Preferences are

$$\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\theta} - 1}{1-\theta}, \quad 0 < \beta < 1, \quad \theta > 1.$$

Goods production is

$$Y_t = K_t^\alpha (\varphi_t H_t)^{1-\alpha}, \quad 0 < \alpha < 1,$$

and human capital evolves according to

$$H_{t+1} = B(1 - \varphi_t)H_t, \quad B > 0.$$

The resource constraint is

$$C_t + K_{t+1} = K_t^\alpha (\varphi_t H_t)^{1-\alpha} + (1 - \delta)K_t, \quad 0 < \delta < 1.$$

Initial stocks $K_0 > 0$ and $H_0 > 0$ are given.

- (a) Set up the social planner's Lagrangian. Derive the first-order conditions with respect to φ_t , H_{t+1} , and K_{t+1} . Write the FOCs in terms of Y_t . [15 marks]
- (b) Eliminate the human-capital multiplier to derive

$$\left(\frac{C_{t+1}}{C_t} \right)^\theta = \beta B \left[\frac{Y_{t+1}}{Y_t} \frac{H_t}{H_{t+1}} \frac{\varphi_t}{\varphi_{t+1}} \right].$$

Then derive the physical-capital Euler equation

$$\left(\frac{C_{t+1}}{C_t} \right)^\theta = \beta \left[\alpha \frac{Y_{t+1}}{K_{t+1}} + (1 - \delta) \right].$$

[13 marks]

- (c) Define a balanced-growth path as an allocation in which C_t , K_t , Y_t , and H_t all grow at the common constant net rate g , while $\varphi_t = \varphi^*$ is constant. Derive

$$1 + g = (\beta B)^{1/\theta},$$

and

$$\varphi^* = 1 - \frac{(\beta B)^{1/\theta}}{B}.$$

State the conditions for positive endogenous growth and an interior time allocation $0 < \varphi^* < 1$. [10 marks]

- (d) Along the balanced-growth path, derive the output-capital ratio Y/K , the capital-output ratio K/Y , and the consumption-output ratio C/Y . Show that

$$\frac{Y}{K} = \frac{B - 1 + \delta}{\alpha}.$$

Then derive the condition under which $C/Y > 0$. [10 marks]

- (e) Let

$$\beta = 0.96, \quad \theta = 2, \quad \alpha = \frac{1}{3}, \quad \delta = 0.05.$$

Compare two education technologies:

$$B_0 = 1.05, \quad B_1 = 1.10.$$

For each case, compute g , φ^* , Y/K , K/Y , and C/Y . Interpret the effect of a higher return to education. [7 marks]